BEURLING-TYPE INVARIANT SUBSPACES OF THE POLETSKY-STESSIN HARDY SPACES IN THE BIDISC

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Dedicated to Prof.Dr. Nazım Sadık on the occasion of his 65th birthday

ABSTRACT. The invariant subspaces of the Hardy space on $H^2(\mathbb{D})$ of the unit disc are very well known however in several variables the structure of the invariant subspaces of the classical Hardy spaces is not yet fully understood. In this study we examine the invariant subspace problem for Poletsky-Stessin Hardy spaces which is a natural generalization of the classical Hardy spaces to hyperconvex domains in $\mathbb{C}^n$. We showed that not all invariant subspaces of $H_{\tilde{u}}^2(\mathbb{D}^2)$ are of Beurling-type. To characterize the Beurling-type invariant subspaces of this space we first generalized the Lax-Halmos theorem of vector valued Hardy spaces to the vector valued Poletsky-Stessin Hardy spaces and then we give a necessary and sufficient condition for the invariant subspaces of $H_{\tilde{u}}^2(\mathbb{D}^2)$ to be of Beurling-type.

INTRODUCTION

In [2], Beurling described all invariant subspaces for the operator of multiplication by z on the Hardy-Hilbert space $H^2(\mathbb{D})$ of the unit disk. In $H^2(\mathbb{D})$, all invariant subspaces are of Beurling-type i.e. they are of the form $fH^2(\mathbb{D})$ where f is an inner function in $H^2(\mathbb{D})$. However, in several variable case the structure of the invariant subspaces cannot be characterized in such a simple form. Although it is quite clear that the Beurling-type subspaces, are invariant; it is known that not all invariant subspaces are of this form. In [7], Jacewicz gave an example of an invariant subspace which can be generated by two functions but cannot be generated by a single function. Later, Rudin [6] gave an example of an invariant subspace which cannot be generated by finitely many elements. There are several studies that try to characterize the Beurling-type invariant subspaces of $H^2(\mathbb{D}^2)$ and in this study we are going to generalize one of these studies given by Sadıkov in [10].

In 2008, Poletsky and Stessin introduced Poletsky-Stessin Hardy spaces and generalized the theory of Hardy spaces to hyperconvex domains in $\mathbb{C}^n$. The structure of these spaces is examined in detail in [8, 11, 16, 13]. Hence, it is natural to ask the invariant subspace problem in the case of Poletsky-Stessin Hardy spaces. In the case of unit disk, Alan and Göğüş [16] showed that all invariant subspaces of the Poletsky-Stessin Hardy space $H_u^2(\mathbb{D})$ are of Beurling-type. In this study we are going to consider the multivariable case for the Poletsky-Stessin Hardy space $H_{\tilde{u}}^2(\mathbb{D}^2)$ of the bidisc. First of all using analogous methods to Jacewicz we will show that there exists an invariant subspace of $H_{\tilde{u}}^2(\mathbb{D}^2)$ which is not of Beurling-type. Then,

we are going to generalize the classical Lax-Halmos theorem to $H_{\tilde{u}}^2(\mathbb{D}^2)$ using the methods of vector valued Hardy spaces. Lastly, we are going to characterize the Beurling-type invariant subspaces of $H_{\tilde{u}}^2(\mathbb{D}^2)$ by generalizing the ideas of Sadıkov [10] that he used for the Beurling-type invariant subspaces of classical Hardy space $H^2(\mathbb{D}^2)$.

1. Preliminaries

In this section we will give the preliminary definitions and some important results that we will use throughout this study. Before proceeding with Poletsky-Stessin Hardy spaces let us first recall the classical Hardy spaces of the polydisc given in [6]:

Definition 1.1. Hardy spaces on the unit polydisc of $\mathbb{C}^n$ are defined for $1 \le p \le \infty$ as :

$$H^p(\mathbb{D}^n) = \{f \in \mathcal{O}(\mathbb{D}^n) : \sup_{0<r<1} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} |f(rz)|^p d\mu\right)^{\frac{1}{p}} < \infty\}$$

where $\mathbb{T}^n$ is torus and μ is the usual product measure on the torus. And

$$H^\infty(\mathbb{D}^n) = \{f \in \mathcal{O}(\mathbb{D}^n) : \sup_{z \in \mathbb{D}^n} |f(z)| < \infty\}$$

In [8], Poletsky & Stessin introduced new Hardy type classes of holomorphic functions on hyperconvex domains Ω in $\mathbb{C}^n$. Before defining these new classes let us first give some preliminary definitions. Let $\varphi : \Omega \to [-\infty, 0)$ be a negative, continuous, plurisubharmonic exhaustion function for Ω. Following [4] we define the pseudoball:

$$B(r) = \{z \in \Omega : \varphi(z) < r\} \quad , \quad r \in [-\infty, 0),$$

and pseudosphere:

$$S(r) = \{z \in \Omega : \varphi(z) = r\} \quad , \quad r \in [-\infty, 0),$$

and set

$$\varphi_r(z) = \max\{\varphi(z), r\} \quad , \quad r \in (-\infty, 0).$$

In [4], Demailly introduced the Monge-Ampère measures in the sense of currents as :

$$\mu_{\varphi,r} = (dd^c \varphi_r)^n - \chi_{\Omega \setminus B(r)} (dd^c \varphi)^n \quad r \in (-\infty, 0).$$

In this study we will use the boundary value characterization of Poletsky-Stessin Hardy spaces in most of the results so let us also mention boundary measures which were introduced by Demailly in [5]. Now let $\varphi : \Omega \to [-\infty, 0)$ be a continuous, plurisubharmonic exhaustion for Ω and suppose that the total Monge-Ampère mass is finite that is, we assume that

$$(1.1) \qquad\qquad MA(\varphi) = \int_\Omega (dd^c \varphi)^n < \infty.$$

Then as r approaches to 0, $\mu_{\varphi,r}$ converges to a positive measure μ_φ weak*-ly on Ω with total mass $\int_\Omega (dd^c \varphi)^n$ and supported on $\partial\Omega$. This measure μ_φ is called the **Monge-Ampère measure on the boundary associated with the exhaustion** φ. As a consequence of ([11], Proposition 2.2.3) we know that the boundary

Monge-Ampère measure $d\mu_u$ is mutually absolutely continuous with respect to the Euclidean measure on the unit circle and we have,

$$(1.2) \qquad d\mu_u = \beta(\theta)d\theta$$

for a positive L^1 function β which is defined as

$$\beta(\theta) = \int_{\mathbb{D}} P(z, e^{i\theta})dd^c u(z).$$

Now we can introduce the Poletsky-Stessin Hardy classes, which will be our main focus throughout this study. In [8], Poletsky & Stessin gave the definition of new Hardy type spaces using Monge-Ampére measures as :

Definition 1.2. $H^p_\varphi(\Omega)$ for $p > 0$, is the space of functions $f \in \mathcal{O}(\Omega)$ such that

$$\limsup_{r \to 0^-} \int_{S_{\varphi,(r)}} |f|^p d\mu_{\varphi,r} < \infty.$$

The norm on these spaces is given by:

$$\|f\|_{H^p_\varphi} = \left(\lim_{r \to 0^-} \int_{S_\varphi(r)} |f|^p d\mu_{\varphi,r} \right)^{\frac{1}{p}}$$

and with respect to these norm the spaces $H^p_\varphi(\Omega)$ are Banach spaces [8].
In Poletsky-Stessin Hardy spaces of the unit disk we have the canonical inner-outer factorization analogous to the classical Hardy space case ([12], Theorem 4.2) but throughout this study we will consider a special type of inner functions which is defined in [16] as follows:

Definition 1.3. Let u be a continuous, subharmonic exhaustion function for $\mathbb{D}$. A function $\phi \in H^2_u(\mathbb{D})$ is a *u-inner* function if $|\phi^*(\xi)|^2 \beta(\xi)$ equals to 1 for almost every $\xi \in \mathbb{T}$ where β is the function given in (1.2).

Remark 1.4. The set of u-inner functions is non-trivial. In fact first all we need to show that there is a holomorphic function ϕ such that $|\phi^*| = \frac{1}{\sqrt{\beta}}$. Now since $\beta(\xi) = \int_{\mathbb{D}} P(z, \xi)dd^c u(z)$ it is a strictly positive function, $\beta(\xi) > c$ for some $c > 0$ so $\frac{1}{\sqrt{\beta}}$ is a bounded, positive function. Then by [6, 3.5.1] we know that we have an analytic function

$$\phi(z) = \exp\left\{ \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \log \frac{1}{\sqrt{\beta}} d\xi \right\}, \quad z \in \mathbb{D}$$

so that $\phi \in H^\infty(\mathbb{D})$ and $|\phi^*| = \frac{1}{\sqrt{\beta}}$ a.e. on $\mathbb{T}$. Then $|\phi^*|^2\beta = 1$ a.e. on $\mathbb{T}$ and $H^\infty(\mathbb{D}) \subset H^2_u(\mathbb{D})$ so $\phi \in H^2_u(\mathbb{D})$.

In the following sections our main focus will be on the Poletsky-Stessin Hardy space, $H^2_{\tilde{u}}(\mathbb{D}^2)$ of the bidisc generated by the following special type of exhaustion function:
Let u be an exhaustion function of the unit disc $\mathbb{D}$ with finite Monge-Ampère mass. Then the following plurisubharmonic function,

$$\tilde{u}(z, w) = \max\{u(z), u(w)\}$$

is an exhaustion for the unit bidisc $\mathbb{D}^2$. For this exhaustion function $\tilde{u}$, the corresponding boundary Monge-Ampère measure on the torus $\mathbb{T}^2$ is given as follows ([11], Theorem 3.2.1) :

$$d\mu_{\tilde{u}}(\theta_1, \theta_2) = d\mu_u(\theta_1)d\mu_u(\theta_2) = \beta(\theta_1)\beta(\theta_2)d\theta_1 d\theta_2.$$

By [6, 3.5.2] we can generalize the definition of u-inner function to $\mathbb{D}^2$ so for a plurisubharmonic exhaustion function $\tilde{u}$ of $\mathbb{D}^2$ we can find a holomorphic function ϕ on $\mathbb{D}^2$ (which is called $\tilde{u}$-inner) such that $\phi \in H^2_{\tilde{u}}(\mathbb{D}^2)$ and $|\phi^*(\xi, \eta)|^2 \beta(\xi)\beta(\eta) = 1$ a.e. on $\mathbb{T}^2$ where $d\mu_{\tilde{u}}(\xi, \eta) = \beta(\xi)\beta(\eta)d\xi d\eta$.

We call M an invariant subspace of $H^2_{\tilde{u}}(\mathbb{D}^2)$ if (a) M is a closed linear subspace of $H^2_{\tilde{u}}(\mathbb{D}^2)$ and (b) $f \in M$ implies $zf \in M$ and $wf \in M$, i.e., multiplication by polynomials maps M into M. An invariant subspace M is called Beurling type if it is of the form $M = \phi H^2(\mathbb{D}^2)$, where ϕ is $\tilde{u}$-inner.

In one variable case Alan and Göğüş [16, Theorem 3.2] extended the classical characterization of invariant subspaces to the Poletsky-Stessin Hardy spaces as follows:

Theorem 1.5. *Let $M \neq \{0\}$ be an invariant subspace of $H^2_u(\mathbb{D})$. Then there exists a u-inner function ϕ so that $M = \phi H^2(\mathbb{D})$.*

2. MAIN RESULTS

In this section we first show that the Poletsky-Stessin Hardy space on the bidisc has an invariant subspaces which is not of the form $fH^2(\mathbb{D}^2)$ for any $f \in H^2_{\tilde{u}}(\mathbb{D}^2)$ in contrast to one variable case. Before proceeding, we recall that $H^2(\mathbb{D}^2)$ can be seen as a closed subspace $H^2(\mathbb{T}^2)$ of the standard Lebesgue space $L^2(\mathbb{T}^2)$ which consists of the functions in $L^2(\mathbb{T}^2)$ with Fourier coefficients vanishing off a pair of nonnegative integers. To each function f in $H^2(\mathbb{T}^2)$ with Fourier series $\sum_{m,n=0}^{\infty} a_{mn}e^{im\theta_1}e^{in\theta_2}$ we associate the function $\sum_{m,n=0}^{\infty} a_{mn}z^m w^n$ analytic on $\mathbb{D}^2$ which we also denote by f. For more details, see [6]. Note that since $H^2_{\tilde{u}}(\mathbb{D}^2)$ is a subspace of $H^2(\mathbb{D}^2)$ by [11, p.54], every function in $H^2_{\tilde{u}}(\mathbb{D}^2)$ also has the Fourier representation above.

Theorem 2.1. *There exists an invariant subspace M of $H^2_{\tilde{u}}(\mathbb{D}^2)$ which is of the form $M = f_1 H^2(\mathbb{D}^2) + f_2 H^2(\mathbb{D}^2)$ for some $f_1, f_2 \in H^2_{\tilde{u}}(\mathbb{D}^2)$ but can not be of the form $M = hH^2(\mathbb{D}^2)$ for any $h \in H^2_{\tilde{u}}(\mathbb{D}^2)$.*

Proof. We choose $f_1(z, w) = \phi(z)\phi(w)q(z)$ and $f_2(z, w) = \phi(z)\phi(w)w$, where ϕ is a non-vanishing u-inner function in $H^2_u(\mathbb{D})$, q is a nonconstant, singular inner function in $H^2(\mathbb{D})$, which means that q never vanishes in $\mathbb{D}$ and has modulus one a.e. on $\mathbb{T}$. It is clear that $f_2 \in H^2_{\tilde{u}}(\mathbb{D}^2)$ and since

$$\begin{aligned}
||f_1||^2_{H^2_{\tilde{u}}(\mathbb{D}^2)} &= \int_{\mathbb{T}}\int_{\mathbb{T}} |\phi^*(z)\phi^*(w)q^*(z)|^2 d\mu_u(z)d\mu_u(w) \\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} |\phi^*(z)|^2|\phi^*(w)|^2|q^*(z)|^2 d\mu_u(z)d\mu_u(w) \\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} |q^*(z)|^2 dz dw \\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} dz dw < \infty,
\end{aligned}$$

$f_1 \in H^2_{\tilde{u}}(\mathbb{D}^2)$. Consider $M = f_1 H^2(\mathbb{D}^2) + f_2 H^2(\mathbb{D}^2)$. It is easily seen that M is an invariant subspace of $H^2_{\tilde{u}}(\mathbb{D}^2)$.

Suppose that M is of the form $M = hH^2(\mathbb{D}^2)$ for any $h \in H^2_{\tilde{u}}(\mathbb{D}^2)$. Let $H^2(S_1)$ denote the subspace of $L^2(\mathbb{T}^2)$ consisting of functions whose Fourier coefficients vanish off the half-plane $S_1 = \{(m,n) \in \mathbb{Z}^2 : m > 0\} \cup \{(0,n) \in \mathbb{Z}^2 : n \geq 0\}$. It is clear that $M_1 := hH^2(S_1)$ is the invariant subspace of $H^2(S_1)$. If q and ϕ have the form $q(z) = \sum_{m=0}^{\infty} a_m z^m$ and $\phi(z) = \sum_{m=0}^{\infty} b_m z^m$ respectively, then we see that

$$b_0^2 a_0 = \phi(z)\phi(w)\left(\sum_{m=0}^{\infty} a_m z^m - \sum_{m=1}^{\infty} a_m z^m w^{-1} w\right) = f_1 - \sum_{m=1}^{\infty} a_m (z^m w^{-1}) f_2$$

lies in M_1. Because $c_m = (m,-1) \in S_1$ for $m \geq 1$ and so $c_m f_2 = z^m w^{-1} f_2 \in M_1$. Since q is a singular function and ϕ is non-vanishing, $b_0^2 a_0 = \phi(0)\phi(0)q(0) \neq 0$, so that the constant functions lie in M_1. Thus $M_1 = hH^2(S_1) = H^2(S_1)$. This property of h for any half-plane containing the support of Fourier transform of h is equivalent to an analytic condition independent of the half-plane [7, p.128]. In particular $hH^2(S_2) = H^2(S_2)$ for the half-plane $S_2 = \{(m,n) \in \mathbb{Z}^2 : n > 0\} \cup \{(m,0) \in \mathbb{Z}^2 : m \geq 0\}$.

Let P be the orthogonal projection of $H^2(S_2)$ onto $H^2_u(\mathbb{D})$ (Remark that the Fourier coefficients of the element of $H^2_u(\mathbb{D})$ are zero for $m < 0$). The invariant subspaces of the form $f_1 H^2(\mathbb{D}^2) + f_2 H^2(\mathbb{D}^2)$ and $hH^2(\mathbb{D}^2)$ are the same. Since S_2 contains the set $\{(m,n) : m \geq 0, n \geq 0\}$, the invariant subspaces $f_1 H^2(S_2) + f_2 H^2(S_2)$ and $hH^2(S_2)$ are the same. These subspaces are denoted by $M_2(f_1, f_2)$ and $M_2(h)$, respectively. $P[M_2(f_1, f_2)]$ is the closed linear span of all $z^m \phi(z)q(z)$, for $m \geq 0$, while $P[M_2(h)] = H^2_u(\mathbb{D})$. Thus by the definition of f_1 from q, it is obtained $\phi(z)q(z)H^2(\mathbb{D}) = H^2_u(\mathbb{D})$. In view of the equality $H^2_u(\mathbb{D}) = \phi(z)H^2(\mathbb{D})$, we have $qH^2(\mathbb{D}) = H^2(\mathbb{D})$, i.e., q is outer in $H^2(\mathbb{D})$. This is contradiction and so M can not be of the form $M = hH^2(\mathbb{D}^2)$ for any $h \in H^2_{\tilde{u}}(\mathbb{D}^2)$. $\qquad\qquad\square$

As a consequence of this theorem, we have that not all invariant subspaces of $H^2_{\tilde{u}}(\mathbb{D}^2)$ are Beurling-type. Then it is natural to ask the structure of Beurling type invariant subspaces of $H^2_{\tilde{u}}(\mathbb{D}^2)$.

First of all, we need to recall the class of vector-valued analytic functions. Let K be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|| \cdot ||_K$. Then by $H^2(K)$ we mean the space of all K-valued holomorphic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ for which the quantity

$$\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{it})\|_K^2 dt = \sum_{n=0}^{\infty} \|a_n\|_K^2 r^{2n}$$

remains bounded for $0 \leq r < 1$. Clearly, $H^2(K)$ is a Hilbert space under the inner product

$$\langle f, g \rangle_2 = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \langle f(re^{i\theta}), g(re^{i\theta}) \rangle_K d\theta = \sum_{n=0}^{\infty} \langle a_n, b_n \rangle_K$$

for any $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in the space. Now if K is a reflexive Banach space then it has Fatou Property i.e. each $f \in H^1(K)$ has non-tangential limits on ∂K ([1, pg:38, 48]). Hence, we know that each $f \in H^2(K)$ has the radial limit f^* as a Bochner measurable function and $f^* \in L^2_+(K)$, where

$L^2_+(K)$ is the space of $L^2(K)$ functions whose negative Fourier coefficients are 0, and we also have $\|f\|_{H^2(K)} = \|f^*\|_{L^2_+(K)}$ (For details see [14, pg:183-186]).

On the other hand, if $B(K, K_1)$ denotes the algebra of all the bounded linear operators from K to K_1, then by $H^\infty(B(K, K_1))$ we mean the algebra of bounded $B(K, K_1)$-valued holomorphic functions Θ on $\mathbb{D}$ in the norm $\|\Theta\|_\infty = \sup_{z \in \mathbb{D}} \|\Theta(z)\|_{B(K,K_1)} < \infty$. It is obvious that each $\Theta \in H^\infty(B(K, K_1))$ gives rise to a bounded linear operator from $H^2(K)$ into $H^2(K_1)$ namely, to an element Θ, we correspond an operator $\hat{\Theta} : H^2(K) \to H^2(K_1)$ that is defined by the formula

$$(\hat{\Theta}f)(z) = \Theta(z)f(z), \, z \in \mathbb{D}, \, f \in H^2(K).$$

An operator-valued $\Theta \in H^\infty(B(K, K_1))$ is called inner if $\Theta(e^{it})$ is an isometry from K into K_1 for almost every t or equivalently, the operator $\hat{\Theta}$ is an isometry.

The reader can find the details of vector-valued analytic functions in [9, 14, 15].

Analogously, we are going to define the vector valued Poletsky-Stessin Hardy spaces as follows:

Definition 2.2. Let K be a Hilbert space, u be a continuous, subharmonic exhaustion function for $\mathbb{D}$. Then the vector valued Poletsky-Stessin Hardy space is defined as follows:

$$H^2_u(K) = \{f : \mathbb{D} \to K, holomorphic : \sup_{r<0} \int_{S_u(r)} \|f(z)\|^2_K d\mu_{u,r}(z) < \infty\}$$

Following step by step the same arguments from the scalar valued case one can easily see that $H^2_u(K) \subset H^2(K)$. Thus, we automatically inherit the radial boundary values from the classical Hardy space $H^2(K)$ and again just rewriting scalar value arguments we have the following boundary value characterization:

Proposition 2.3. Let $f \in H^2_u(K)$ and f^* be its radial boundary value function. Then

$$\|f\|^2_{H^2_u(K)} = \|f^*\|^2_{L^2_{+,u}(K)} = \int_{\mathbb{T}} \|f^*(\xi)\|^2_K d\mu_u(\xi)$$

Proof. Directly follows from the scalar valued argument given in ([11], Theorem 2.2.1). □

Now, recall the Wold decomposition for isometries [14, p.3, Theorem 1.1]: Let V be an arbitrary isometry on a Hilbert space H. Then H decomposes into an orthogonal sum $H = H_1 \oplus H_2$ such that H_1 and H_2 reduce V, the part of V on H_1 is unitary and the part of V on H_2 is a unilateral shift. This decomposition is uniquely determined, indeed we have

$$H_1 = \bigcap_{n=0}^{\infty} V^n H \text{ and } H_2 = \bigoplus_{n=0}^{\infty} V^n E \text{ where } E = H \ominus VH.$$

The space H_1 or H_2 may be absent, i.e., equal to $\{0\}$.

If the Poletsky-Stessin Hardy space over the bidisc is interpreted as the vector-valued analytic functions on the unit disc of complex plane, then invariant subspaces under the multiplication operator by the independent variable z are described in terms of Lax-Halmos theorem.

Theorem 2.4. *Let M be a non-zero subspace of $H_u^2(H_u^2(\mathbb{D}))$. M is invariant under the multiplication operator by the independent variable z if and only if there exists a Hilbert space E and an inner function $\Theta \in H^\infty(B(E, \varphi H_u^2(\mathbb{D})))$ such that $M = \hat{\Theta} H^2(E)$. This class of the functions Θ is denoted by $\{\Theta_M\}$.*

First of all, we need the following lemma:

Lemma 2.5. *$H_u^2(H_u^2(\mathbb{D})) = \varphi H^2(H_u^2(\mathbb{D}))$ where φ is the u-inner function which gives $H_u^2(\mathbb{D}) = \varphi H^2(\mathbb{D})$.*

Proof. Let $f \in \varphi H^2(H_u^2(\mathbb{D}))$. Then $f(z) = \varphi(z)h(z)$ where $h(z) \in H_u^2(\mathbb{D})$. Now

$$\int_{\mathbb{T}} ||f(z)||^2_{H_u^2(\mathbb{D})} d\mu_u(z) = \int_{\mathbb{T}} |\varphi(z)|^2 ||h(z)||^2_{H_u^2(\mathbb{D})} d\mu_u(z)$$

$$= \int_{\mathbb{T}} ||h(z)||^2_{H_u^2(\mathbb{D})} d\theta = ||h||_{H^2(H_u^2(\mathbb{D}))} < \infty.$$

Then $f \in H^2(H_u^2(\mathbb{D}))$ and $H_u^2(H_u^2(\mathbb{D})) \supseteq \varphi H^2(H_u^2(\mathbb{D}))$. Conversely, let $f \in H_u^2(H_u^2(\mathbb{D}))$. Then consider the function $\dfrac{f(z)}{\varphi(z)}$. We want to show that $\dfrac{f(z)}{\varphi(z)} = h(z)$ is in $H^2(H_u^2(\mathbb{D}))$. First of all for all $z \in \mathbb{D}$ $h(z) = \dfrac{f(z)}{\varphi(z)} = \dfrac{h_z}{\varphi(z)} \in H_u^2(\mathbb{D})$ since $h_z \in H_u^2(\mathbb{D})$ and

$$\int_{\mathbb{T}} ||h(z)||^2_{H_u^2(\mathbb{D})} d\theta = \int_{\mathbb{T}} \frac{1}{|\varphi(z)|^2} \int_{\mathbb{D}} |h_z(w)|^2 d\mu_u(w) d\theta$$

$$= \int_{\mathbb{D}} \int_{\mathbb{D}} |h_z(w)|^2 d\mu_u(w) d\mu_u(z)$$

$$= \int_{\mathbb{T}} ||f(z)||^2_{H_u^2(\mathbb{D})} d\mu_u(z) = ||f|| < \infty.$$

since $|\varphi|^2 \beta = 1$ a.e. we have $|\varphi|^2 d\mu_u = d\theta$. Hence we obtain $H_u^2(H_u^2(\mathbb{D})) \subseteq \varphi H^2(H_u^2(\mathbb{D}))$. $\qquad \square$

Proof of Theorem (2.4). If $\Theta \in H^\infty(B(E, \varphi H_u^2(\mathbb{D})))$ is an inner function then the corresponding operator is isometric and hence $M = \hat{\Theta} H^2(E)$ is closed. Its invariance for the multiplication by z is obvious.

Now let M be an invariant subspace of $H_u^2(H_u^2(\mathbb{D}))$ under multiplication by z. Now first of all embedding $H_u^2(\mathbb{D})$ in $H^2(H_u^2(\mathbb{D}))$ as a subspace by identifying the element $\lambda \in H_u^2(\mathbb{D})$ with the constant function $\lambda(z) = \lambda$; $H_u^2(\mathbb{D})$ is then wandering for the multiplication operator by z and

$$H^2(H_u^2(\mathbb{D})) = \bigoplus_{n=0}^{\infty} z^n H_u^2(\mathbb{D})$$

and by Lemma (2.5) we have

$$H_u^2(H_u^2(\mathbb{D})) = \varphi H^2(H_u^2(\mathbb{D})) = \bigoplus_{n=0}^{\infty} z^n (\varphi H_u^2(\mathbb{D})).$$

Let V denote the restriction of the multiplication operator by z to the invariant subspace M; this is an isometry on M. We have

$$\bigcap_{n=0}^{\infty} V^n M \subset \bigcap_{n=0}^{\infty} z^n H_u^2(H_u^2(\mathbb{D})) \subset \bigcap_{n=0}^{\infty} z^n H^2(H_u^2(\mathbb{D})) = \{0\}$$

and thus V has no unitary part so that the corresponding Wold-decomposition of the form $M = \bigoplus_{n=0}^{\infty} V^n E$, where $E = M \ominus (VM)$. Let us now apply Lemma 3.2 [14, p.195] to $R_+ = M$, $U_+ = V$, $U = E$, $R'_+ = H_u^2(H_u^2(\mathbb{D}))$, $U_+ = $ multiplication by z, $U' = \varphi H_u^2(\mathbb{D})$ and $Q = $ the identical transformation of M into $H_u^2(H_u^2(\mathbb{D}))$, then there exists an inner function $\Theta \in H^{\infty}(B(E, \varphi H_u^2(\mathbb{D})))$ such that

$$(2.1) \qquad\qquad \phi_+^{\varphi H_u^2(\mathbb{D})} Q = \hat{\Theta} \phi_+^E$$

on M. Since $\varphi H_u^2(\mathbb{D})$ consists of the constant functions in $H_u^2(H_u^2(\mathbb{D}))$, the Fourier representation of $H_u^2(H_u^2(\mathbb{D}))$ with respect to multiplication by z is identity transformation. On the other hand we have $Qh = h$ for $h \in M$. Thus (2.1) reduces to $h = \hat{\Theta} \phi_+^E h$, $h \in M$ and hence we have $M = \hat{\Theta} \phi_+^E M = \hat{\Theta} H^2(E)$ as claimed.   □

Lemma 2.6. *Vector valued Poletsky-Stessin Hardy space $H_u^2(H_u^2(\mathbb{D}))$ is isometrically isomorphic to the Poletsky-Stessin Hardy space $H_{\tilde{u}}^2(\mathbb{D}^2)$ of bidisc.*

Proof. Let $\tilde{u}(z, w) = \max\{u(z), u(w)\}$ be the exhaustion function for the bidisc $\mathbb{D}^2$ then we have the following isometric isomorphism between the Banach spaces $H_u^2(H_u^2(\mathbb{D}))$ and $H_{\tilde{u}}^2(\mathbb{D}^2)$:
Take $g \in H_u^2(H_u^2(\mathbb{D}))$ then $g(z) = g_z(w)$ for some $g_z \in H_u^2(\mathbb{D})$. Now consider the corresponding function $\bar{g}$ on $\mathbb{D}^2$ defined as $\bar{g}(z, w) = g_z(w)$ then using ([11], Theorem 3.2.1) we have,

$$\begin{aligned}
\|\bar{g}\|_{H_{\tilde{u}}^2(\mathbb{D}^2)}^2 &= \int_{\mathbb{T}^2} |\bar{g}^*(\xi, \eta)|^2 d\mu_{\tilde{u}}(\xi, \eta) \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} |\bar{g}^*(\xi, \eta)|^2 d\mu_u(\eta) d\mu_u(\xi) \\
&= \int_{\mathbb{T}} \|g_z\|_{H_u^2(\mathbb{D})}^2 d\mu_u = \|g\|_{H_u^2(H_u^2(\mathbb{D}))}^2.
\end{aligned}$$

□

Suppose that a subspace M of $H_{\tilde{u}}^2(\mathbb{D}^2)$ which is invariant under the multiplication operators by independent variables z and w is of Beurling-type, i.e., M is of the form $M = \phi H^2(\mathbb{D}^2)$ for some u-inner function ϕ. Since M is invariant under the multiplication by z, in view of Lemma (2.6) and Theorem (2.4), it can be described by the class of functions $\{\Theta_M\}$. However, the subspaces determined by these class of functions $\{\Theta_M\}$ are not generally of Beurling-type and the following theorem gives a condition for those subspaces which are defined by $\{\Theta_M\}$ to be of Beurling-type using the simple relation $H^2(H_u^2(\mathbb{D})) = H^2(N) \oplus H^2(N^{\perp})$, where N is a subspace of $H_u^2(\mathbb{D})$ and $N^{\perp}$ its complement.

Theorem 2.7. *A subspace M of $H_{\tilde{u}}^2(\mathbb{D}^2)$ is invariant under the multiplication operators by the independent variables z and w is Beurling-type if and only if there exists at least an operator valued holomorphic function $\Theta(z)$, $z \in \mathbb{D}$ in the class $\{\Theta_M\}$ such that for every $z_0 \in \mathbb{D}$ the operator $\Theta(z_0)$ on $H_u^2(\mathbb{D})$ commutes with the multiplication operator by w in $H_u^2(\mathbb{D})$.*

Before starting the proof of the main theorem, we need the following preliminary results ([13], pp:34):
Define

$$\tilde{\alpha}(z) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(\beta(e^{i\theta})) d\theta$$

then the function

$$A(z) = e^{\tilde{\alpha}(z)}$$

is a holomorphic function of the unit disc which extends smoothly to the unit circle with the property $|A^*(e^{i\theta})| = \beta(e^{i\theta})$.

Theorem 2.8 (Shresta'15). *The space $H_u^p(\mathbb{D})$ is isometrically isomorphic to $H^p(\mathbb{D})$.*

Remark 2.9. The above mentioned isomorphism is given as follows:

$$H_u^p(\mathbb{D}) \leftrightarrow H^p(\mathbb{D})$$

$$f \leftrightarrow A^{\frac{1}{p}} f$$

Lemma 2.10. *The set* $\left\{ \dfrac{z^n w^m}{\sqrt{A(z)}\sqrt{A(w)}} \right\}_{n,m \geq 0}$ *is an orthonormal basis of $H_{\tilde{u}}(\mathbb{D}^2)$.*

Proof. It is enough to show that $\left\{ \dfrac{z^n w^m}{\sqrt{A(z)}\sqrt{A(w)}} \right\}_{n,m \geq 0}$ is a complete orthonormal set in $H_{\tilde{u}}(\mathbb{D}^2)$ by [3, Theorem 4.13, p.16]. First we will show the orthonormality of this set:

$$\left\| \frac{z^n w^m}{\sqrt{A(z)}\sqrt{A(w)}} \right\|_{H_{\tilde{u}}(\mathbb{D}^2)} = \left\langle \frac{z^n w^m}{\sqrt{A(z)}\sqrt{A(w)}}, \frac{z^n w^m}{\sqrt{A(z)}\sqrt{A(w)}} \right\rangle$$

$$= \int_{\mathbb{T}^2} \frac{e^{in\theta_1} e^{im\theta_2} e^{-in\theta_1} e^{-im\theta_2}}{|A^*(e^{i\theta_1})||A^*(e^{i\theta_2})|} d\mu_{\tilde{u}}$$

$$= \left(\int_{\mathbb{T}} \frac{d\mu_u(\theta_1)}{|A^*(e^{i\theta_1})|} \right) \left(\int_{\mathbb{T}} \frac{d\mu_u(\theta_2)}{|A^*(e^{i\theta_2})|} \right) = \int_{\mathbb{T}} d\theta_1 \int_{\mathbb{T}} d\theta_2 = 1$$

For any $(n_1, m_1), (n_2, m_2)$ such that $(n_1, m_1) \neq (n_2, m_2)$,

$$\left\langle \frac{z_1^n w_1^m}{\sqrt{A(z)}\sqrt{A(w)}}, \frac{z_2^n w_2^m}{\sqrt{A(z)}\sqrt{A(w)}} \right\rangle$$

$$= \left(\int_{\mathbb{T}} \frac{e^{i(n_1 - n_2)\theta_1}}{|A^*(e^{i\theta_1})|} d\mu_u(\theta_1) \right) \left(\int_{\mathbb{T}} \frac{e^{i(m_1 - m_2)\theta_2}}{|A^*(e^{i\theta_2})|} d\mu_u(\theta_2) \right)$$

$$= \left(\int_{\mathbb{T}} e^{i((n_1 - n_2)\theta_1)} d\theta_1 \right) \left(\int_{\mathbb{T}} e^{i((m_1 - m_2)\theta_2)} d\theta_2 \right) = 0$$

since $\{z^k\}_{k \geq 0}$ is orthonormal in $H^2(\mathbb{D})$.

As for completeness let $f(z, w) \in H_{\tilde{u}}(\mathbb{D}^2)$ be such that

$$\int_{\mathbb{T}^2} f(e^{i\theta_1}, e^{i\theta_2}) \frac{e^{-in\theta_1} e^{-im\theta_2}}{\sqrt{\overline{A^*(e^{i\theta_1})}}\sqrt{\overline{A^*(e^{i\theta_2})}}} d\mu_u(\theta_1) d\mu_u(\theta_2) = 0$$

for all n, m. We claim that $f \equiv 0$. We have, by Fubini's theorem,

$$0 = \int_{\mathbb{T}^2} f(e^{i\theta_1}, e^{i\theta_2}) \frac{e^{-in\theta_1} e^{-im\theta_2}}{\sqrt{\overline{A^*(e^{i\theta_1})}}\sqrt{\overline{A^*(e^{i\theta_2})}}} d\mu_u(\theta_1) d\mu_u(\theta_2)$$

$$= \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f(e^{i\theta_1}, e^{i\theta_2}) \frac{e^{-in\theta_1}}{\sqrt{\overline{A^*(e^{i\theta_1})}}} d\mu_u(\theta_1) \right) \frac{e^{-im\theta_2}}{\sqrt{\overline{A^*(e^{i\theta_2})}}} d\mu_u(\theta_2)$$

Now, using the fact that $\{\frac{w^m}{\sqrt{A(w)}}\}$ is an orthonormal basis for $H_u^2(\mathbb{D})$, we have that for all n

$$\int_{\mathbb{T}} f(e^{i\theta_1}, e^{i\theta_2}) \frac{e^{-in\theta_1}}{\sqrt{A^*(e^{i\theta_1})}} d\mu_u(\theta_1) = 0 \ \ \mu_u - a.e.$$

Let $E_n \subset \mathbb{T}$ be the set of measure zero where the above equality does not hold and let $E = \bigcup_n E_n$. Then for $e^{i\theta_2} \notin E$,

$$\int_{\mathbb{T}} f(e^{i\theta_1}, e^{i\theta_2}) \frac{e^{-in\theta_1}}{\sqrt{A^*(e^{i\theta_1})}} d\mu_u(\theta_1) = 0$$

for all n, and thus again using the fact that $\{\frac{z^n}{\sqrt{A(z)}}\}$ is a complete orthonormal set in $H_u^2(\mathbb{D})$, we have that $f(e^{i\theta_1}, e^{i\theta_2}) = 0$ μ_u-a.e. Therefore $f = 0$, $\mu_{\tilde{u}}$-a.e. which gives $f \equiv 0$. Hence the claim follows. $\square$

Lemma 2.11. *The set of all bounded linear operators on $H_u^2(\mathbb{D})$ commuting with the operators of multiplication by the independent variable z is the set of all multiplication operators by multipliers in $H^\infty(\mathbb{D})$.*

Proof. The claim is clear since the commutant of the multiplication operator by independent variable on $H^2(\mathbb{D})$ is the set of all multiplication operators by multipliers in $H^\infty(\mathbb{D})$ by [17, Problem 116] and $H_u^2(\mathbb{D})$ is subspace of $H^2(\mathbb{D})$. $\square$

Proof of Theorem (2.7). Suppose that there is a Θ in the class $\{\Theta_M\}$ such that for any fixed $z_0 \in \mathbb{D}$, $\Theta(z_0)$ commutes with the multiplication operator by w in $H_u^2(\mathbb{D})$. Since, by Lemma (2.11), the commutant of the multiplication operator by w in $H_u^2(\mathbb{D})$ is $H^\infty(\mathbb{D})$, it follows that $\Theta(z_0) \in H^\infty(\mathbb{D})$ for every $z_0 \in \mathbb{D}$. Let's note that the function $z \to \Theta(z)1$, where the function 1 in $H_u^2(\mathbb{D})$ is identically equal to 1, is an analytic function of z taking values in $H_u^2(\mathbb{D})$. Hence it follows that if $\phi = \Theta 1$, then $\Theta(z_0)1$ coincides with a function $\phi(z_0, w)$, and the family of functions $w \to \Theta(z_0)(w)$, $w \in \mathbb{D}$ is a family generated by an analytic function ϕ. To obtain that ϕ is a $\tilde{u}$-inner function it is enough to show that the multiplication operator by ϕ in $H_{\tilde{u}}^2(\mathbb{D}^2)$ is an isometry. If $g \in H_{\tilde{u}}^2(\mathbb{D}^2)$, then for $g(z, w) = g_z(w)$, by [11], we have

$$||g||^2 = \int_{\mathbb{T}} \int_{\mathbb{T}} |g^*(\xi, \eta)|^2 d\mu_u(\xi) d\mu_u(\eta) = \int_{\mathbb{T}} ||g_\xi||^2 d\mu_u(\eta).$$

Applying this to the function $\phi\varphi$ we obtain

$$||\phi\varphi||^2 = \int_{\mathbb{T}} ||\phi_\eta \varphi_\eta||^2 d\mu_u(\eta)$$

and by assumption ϕ_η is an isometric operator for almost all η, therefore $||\phi_\eta \varphi_\eta|| = ||\varphi_\eta||$ for almost all η and $||\phi\varphi||^2 = ||\varphi||^2$. Thus the operator Θ in $H_u^2(H_u^2(\mathbb{D}))$ and the multiplication operator by $\phi = \Theta 1$ in $H_{\tilde{u}}^2(\mathbb{D}^2)$ are bounded operators which agree on vectors the type $\frac{z^k w^l}{\sqrt{A(z)}\sqrt{A(w)}}$, $k, l \geq 0$ under the canonical isomorphism between $H_u^2(H_u^2(\mathbb{D}))$ and $H_{\tilde{u}}^2(\mathbb{D}^2)$. In Lemma (2.10) we have proved that the elements $\frac{z^k w^l}{\sqrt{A(z)}\sqrt{A(w)}}$, $k, l \geq 0$ are dense in $H_{\tilde{u}}^2(\mathbb{D}^2)$. Hence $\phi = \Theta 1$ and Θ correspond to each other.

For the converse direction now suppose that M is a subspace generated by a $\tilde{u}$-inner function ϕ then for almost any $\xi \in \mathbb{T}$, $\phi^*(\xi, \cdot)$ is a u-inner function in $H^\infty(\mathbb{D})$ and the radial boundary values of the operator valued function $\Theta(z)$, where $\Theta(z)$ is the operator of multiplication by the function ϕ, is an isometry almost everywhere. Hence the result follows. $\qquad\square$

References

[1] Aytuna, A., Some results on H^p-Spaces on strictly Pseudoconvex Domains, PhD Dissertation, University of Washington, 1976.

[2] Beurling, A., On two problems concerning linear transformations in Hilbert space. Acta Math. 81, (1948), 17 pp.

[3] Conway, J.B., A course in functional analysis. Second edition, Springer-Verlag, New York, 1990.

[4] Demailly, J.P., Mesures de Monge-Ampère et Caractérisation Géométrique des Variétés Algébraiques Affines. Mémoire de la Société Mathématique de France **19**,1-124, (1985).

[5] Demailly, J.P., Mesures de Monge-Ampère et Mesures Pluriharmoniques. Matematische Zeitschrift, No:194, 519-564, (1987).

[6] Rudin, W., Function theory in polydiscs. W. A. Benjamin, Inc., New York-Amsterdam 1969 vii+188 pp.

[7] Jacewicz, C. A., A nonprincipal invariant subspace of the Hardy space on the torus. Proc. Amer. Math. Soc. 31 (1972), 127-129.

[8] Evgeny A. Poletsky,Michael I. Stessin, *Hardy and Bergman Spaces on Hyperconvex Domains and Their Composition Operators* Indiana Univ. Math. J. **57**, 2153-2201, (2008).

[9] Radjavi, H., Rosenthal, P., Invariant subspaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77. Springer-Verlag, New York-Heidelberg, 1973. xi+219 pp.

[10] Sadikov, N. M. Invariant subspaces in the Hardy space on a bidisk. (Russian) Spectral theory of operators and its applications, No. 7 (Russian), 186200, "lm", Baku, 1986.

[11] Şahin, S., Monge-Ampère Measures and Poletsky-Stessin Hardy Spaces on Bounded Hyperconvex Domains, PhD Dissertation, Sabancı University, 2014.

[12] Şahin, S., Poletsky-Stessin Hardy Spaces on Domains Bounded by An Analytic Jordan Curve in $\mathbb{C}$. Complex Variables and Elliptic Equations,DOI:10.1080/17476933.2014.1001112, (2014).

[13] Shresta, K., Poletsky-Stessin Hardy Spaces on the Unit Disk, PhD Dissertation, Syracuse University, Dissertations-ALL. Paper 279, (2015).

[14] B. Sz.-Nagy and C. Foias, Harmonic Analysis of Operators on Hilbert Space, Akademiai Kiad, Budapest, 1970.

[15] Rosenblum, M.; Rovnyak, J. Hardy classes and operator theory. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1985. xiv+161 pp.

[16] Alan, M.A., Göğüş, N.G., Poletsky-Stessin Hardy Spaces in the Plane. Complex Analysis and Operator Theory, DOI:10.1007/s11785-013-0334-2.

[17] Halmos, Paul Richard A Hilbert space problem book. Second edition. Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1982. xvii+369

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